

Supersymmetric QCD in the Noncommutative Geometry

Satoshi Ishihara,^{*} Hironobu Kataoka,[†] Atsuko Matsukawa,[‡] and Hikaru Sato[§]
Department of Physics, Hyogo University of Education
Shimokume, Kato-shi, Hyogo 673-1494, Japan

Masafumi Shimojo[¶]
Department of Electronics and Information Technology
Fukui National College of Technology
Geshi-cho, Sabae-shi, Fukui 916-8507, Japan
(Dated: December 9, 2011)

Introduction of supersymmetry into the noncommutative geometry is investigated. We propose a new Dirac operator which plays the role of the metric over the extended algebra of chiral and antichiral supermultiplets and is invariant under the supersymmetry transformations. Inner automorphisms for the algebra generate vector supermultiplets as an internal fluctuation of the metric. We show that the supersymmetric QCD action for these supermultiplets is correctly given by the spectral action principle.

PACS numbers: 11.10.Nx, 11.30.Pb, 12.10.-g, 12.60.Jv

I. INTRODUCTION

An application of noncommutative geometry (NCG) to particle physics was initiated by Connes [1] and subsequent works followed [2–5]. This approach made possible full geometric description of the interaction of elementary particles [6, 7]. Following the scheme of NCG, the noncommutative geometric standard model was studied by various authors [8–11]. A remarkable feature of this geometric approach is that both the gauge fields and the Higgs fields are introduced by internal fluctuations of the metric of NCG. In the traditional spontaneously broken gauge theory, the Higgs Lagrangian contains many free parameters so that they spoil the predictive power of the model. In the NCG models, the Higgs field emerges on the same footing as the gauge field and the Higgs coupling constants are related to the gauge coupling constants.

The NCG standard model, extended to include neutrino masses, was constructed [12, 13]. Although the standard model of elementary particles provides a remarkably successful description of presently known phenomena, there are some difficulties and they suggest the existence of new physics beyond the standard model. One of them is that the gauge coupling unification in the renormalization group equation is not viable phenomenologically in the framework of the minimal Standard Model. In addition, there is the infamous “hierarchy problem”, which is that the Higgs squared mass parameter m_H^2 receives enormous quantum corrections from the virtual effects of every particle that couples to the Higgs field. These difficulties also exist in the NCG standard

model.

It is known that these shortcomings are remedied by introducing supersymmetry into the standard model [14]. In order for the NCG standard model to be phenomenologically viable, it is quite desirable to incorporate supersymmetry into the model. The purpose of this paper is to investigate how to introduce supersymmetry in NCG. Following our prescription of supersymmetric NCG we shall show that the supersymmetric QCD action is correctly derived.

The fundamental aspect of supersymmetry in NCG was discussed by Connes [1]. Subsequently the early attempts to incorporate supersymmetry in NCG were made by several authors [15–18]. Recently the supersymmetric QCD was derived in [19] on the basis of the spectral action principle [20]. In their works the superpartners of the QCD-particles were added in such a way that fermions are elements of the Hilbert space while bosons arise as inner fluctuations of a Dirac operator. It seems rather unnatural, however, that the fermions and bosons in the same supermultiplet have the different origin.

Our approach is different from those previous works as outlined below. The basic element of NCG consists of an involutive algebra of operators and of a selfadjoint Dirac operator in the Hilbert space. These ingredients of NCG are formulated in the Euclidean space-time. In order to incorporate supersymmetry, however, we must work in the Minkowskian space-time. Moreover, we enlarge the Hilbert space and the algebra in such a way that they include both fermions and bosons which form a supermultiplet. In this situation we construct a new generalized supersymmetric Dirac operator, and show that the supersymmetric action is derived correctly from the generalized supersymmetric Dirac operator by the use of the spectral action formula. In order to check the axioms of NCG [21], we change the signature from Minkowskian to Euclidean by the Wick rotation, since the axioms of NCG have been stated in the Euclidean space-time.

^{*} satoshi@yukawa.kyoto-u.ac.jp

[†] hrkataoka@hyogo-c.ed.jp

[‡] matsukawa@cap.ocn.ne.jp

[§] hikaru_sato@gakushikai.jp

[¶] shimo0@ei.fukui-nct.ac.jp

Supersymmetric extension of the gauge symmetry is derived from inner automorphisms for the algebra of supermultiplets. We shall see that the effect of inner automorphisms on the metric gives rise to internal fluctuations and replace the supersymmetric Dirac operator by the modified one. This mechanism generates exactly the gauge supermultiplets. Then we can take a step toward the construction of the supersymmetric QCD in our geometric approach. It is shown that the supersymmetric QCD action is derived on the basis of the spectral action principle. For the sake of simplicity we consider in this paper the flat space without the gravity couplings.

The present paper is organized as follows: In section 2, we consider a functional space \mathcal{H}_M which consists of spinor and scalar functions on the Minkowskian space-time. The supersymmetry transformation is imposed on these functions. We also introduce an algebra \mathcal{A}_M , whose elements are chiral and antichiral supermultiplets under the supersymmetry transformation. Then we define the supersymmetric operator \mathcal{D}_M in \mathcal{H}_M , which turns out to be the generalized Dirac operator if we change the signature from the Minkowskian to the Euclidean one.

In section 3, we show that the $(\mathcal{A}_M, \mathcal{H}_M, \mathcal{D}_M)$ is the spectral triple of the Euclidean NCG. In the Euclidean signature, \mathcal{H}_M turns out to be the Hilbert space and \mathcal{D}_M is the Dirac operator with the compact resolvent. We can define the $Z/2$ grading of \mathcal{H}_M , which corresponds to the chirality of the supersymmetry transformation.

In section 4, the vector supermultiplet is introduced as the internal fluctuation on the metric with respect to the supersymmetric Dirac operator \mathcal{D}_M . The vector fields as well as their superpartners are constructed as the bilinear form of the functions in the chiral and antichiral supermultiplets. We derive the modified supersymmetric Dirac operator which incorporates the vector supermultiplet.

In section 5, internal degrees of freedom are incorporated by introducing finite space with the spectral triple $(\mathcal{H}_F, \mathcal{A}_F, \mathcal{D}_F)$. In this paper we consider quarks and color degrees of freedom by letting \mathcal{A}_F be the algebra of 3×3 complex matrices and \mathcal{H}_F be the Hilbert space with the basis of internal degrees of freedom of quarks.

In section 6, we calculate the supersymmetric QCD action by applying the spectral action principle. Since the supersymmetric Dirac operator \mathcal{D}_M is expressed in the Minkowskian signature, we change the signature to the Euclidean one in order to calculate the spectral action using the formula for the operator of Laplace type. After the Wick rotation back to Minkowskian signature, we can finally obtain the supersymmetric QCD action integral.

Finally, in section 7 we give our conclusions and outlook.

II. SUPERSYMMETRY AND NONCOMMUTATIVE GEOMETRY

The basic element of NCG consists of an involutive algebra \mathcal{A} of operators in Hilbert space \mathcal{H} and of a self-adjoint unbounded operator \mathcal{D} called the Dirac operator in \mathcal{H} . A set of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is named a spectral triple or a K -cycle [1]. If we consider a flat space, the space-time manifold of NCG is Euclidean. In order to introduce supersymmetry, however, we must work in the Minkowskian space-time M . So, our strategy is the following: First we consider $(\mathcal{A}_M, \mathcal{H}_M, \mathcal{D}_M)$ on the Minkowskian space-time manifold M , and introduce there the supersymmetry degrees of freedom. Then we relate it to the spectral triple of the Euclidean NCG by passing from the Minkowskian signature $g^{\mu\nu} = (-1, 1, 1, 1)$ to the Euclidean signature $\eta^{\mu\nu} = (1, 1, 1, 1)$ by the Wick rotation, $t \rightarrow it$. When we calculate the spectral action, we work in the Euclidean signature and then come back to the the Minkowskian signature to obtain the physical results.

In order to introduce supersymmetry we prepare the functional space \mathcal{H}_M with spinor and scalar functions of $C^\infty(M)$. We consider the subsets \mathcal{H}_+ and \mathcal{H}_- of \mathcal{H}_M . The elements of \mathcal{H}_+ are denoted by

$$(\Psi_+)_i = (\varphi_+(x), \psi_{+\alpha}(x), F_+(x)), \quad (2.1)$$

in the vector notation, $i = 1, 2, 3$. Here $\varphi_+(x)$ and $F_+(x)$ are complex scalar functions with mass dimension 1 and 2, respectively, and $\psi_{+\alpha}(x)$, $\alpha = 1, 2$ are the Weyl spinors on the space-time manifold M which have mass dimension $\frac{3}{2}$.

The functions in Eq.(2.1) are assumed to obey the following supersymmetry transformation of the chiral supermultiplet:

$$\begin{cases} \delta_\xi \varphi_+ &= \sqrt{2} \xi^\alpha \psi_{+\alpha}, \\ \delta_\xi \psi_{+\alpha} &= i\sqrt{2} \sigma_{\alpha\dot{\alpha}}^\mu \bar{\xi}^{\dot{\alpha}} \partial_\mu \varphi_+ + \sqrt{2} \xi_\alpha F_+, \\ \delta_\xi F_+ &= i\sqrt{2} \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_{+\alpha}. \end{cases} \quad (2.2)$$

The element of \mathcal{H}_- is denoted by the vector with the index $\bar{i} = 1, 2, 3$, and is the following antichiral supermultiplet,

$$(\Psi_-)_{\bar{i}} = (\varphi_-(x), \psi_{-\dot{\alpha}}(x), F_-(x)). \quad (2.3)$$

The functions in Ψ_- obey the antichiral supersymmetry transformation:

$$\begin{cases} \delta_\xi \varphi_- &= \sqrt{2} \bar{\xi}_{\dot{\alpha}} \psi_{-\dot{\alpha}}, \\ \delta_\xi \psi_{-\dot{\alpha}} &= i\sqrt{2} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \xi_\alpha \partial_\mu \varphi_- + \sqrt{2} \bar{\xi}^{\dot{\alpha}} F_-, \\ \delta_\xi F_- &= i\sqrt{2} \xi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \psi_{-\dot{\alpha}}. \end{cases} \quad (2.4)$$

Weyl spinors with undotted indices α , ($\alpha = 1, 2$) transform as the $(\frac{1}{2}, 0)$ representation of the Lorentz group, $SL(2, C)$, while those with dotted indices $\dot{\alpha}$, ($\dot{\alpha} = 1, 2$) transform as the $(0, \frac{1}{2})$ representation. The indices α are raised and lowered with the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$, where $\varepsilon^{12} = \varepsilon_{21} = 1$. The same holds for dotted

indices $\dot{\alpha}$. The ε -tensor is also used to raise the indices of the σ -matrices as $\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\beta}\varepsilon^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\beta}}$. Here and in what follows we use the notation and convention of [22]. We assume that the functions of chiral and antichiral supermultiplets and their derivatives are $C^\infty(M)$.

The algebra in the space $\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_-$ is now defined by $\mathcal{A}_M = \mathcal{A}_+ \oplus \mathcal{A}_-$. The element of \mathcal{A}_+ is given in the following matrix form;

$$(u_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi_a & 0 & 0 \\ \psi_{a\alpha} & \varphi_a & 0 \\ F_a & -\psi_a^\alpha & \varphi_a \end{pmatrix} \in \mathcal{A}_+, \quad (2.5)$$

where $\varphi_a, \psi_{a\alpha}$ and F_a form a chiral supermultiplet and they obey the supersymmetry transformation (2.2). In Eq.(2.5), m_0 stands for the mass parameter which was inserted to adjust the mass dimension. The multiplication rule among the elements in \mathcal{A}_+ is now given by

$$(u_3)_{ik} = (u_1)_{ij}(u_2)_{jk}, \quad (2.6)$$

$$\begin{cases} \varphi_3 &= \varphi_1\varphi_2/m_0, \\ \psi_{3\alpha} &= (\psi_{1\alpha}\varphi_2 + \varphi_1\psi_{2\alpha})/m_0, \\ F_3 &= (\varphi_1F_2 + F_1\varphi_2 - \psi_1^\alpha\psi_{2\alpha})/m_0, \end{cases} \quad (2.7)$$

where $\varphi_3, \psi_{3\alpha}$ and F_3 transform again as Eq.(2.2) and belong to \mathcal{A}_+ . In Eq.(2.6) and in what follows, repeated matrix indices should be understood to take summation over the matrix size. Complex conjugate element of Eq.(2.5) belongs to \mathcal{A}_- and is given by

$$(\bar{u}_a)_{\bar{i}\bar{j}} = \frac{1}{m_0} \begin{pmatrix} \varphi_a^* & 0 & 0 \\ \bar{\psi}_a^{\dot{\alpha}} & \varphi_a^* & 0 \\ F_a^* & -\bar{\psi}_{a\dot{\alpha}} & \varphi_a^* \end{pmatrix} \in \mathcal{A}_-, \quad (2.8)$$

which obeys the supersymmetry transformation for the antichiral supermultiplet given by Eq.(2.4).

The algebra \mathcal{A}_+ is represented in the space \mathcal{H}_+ by the following formula,

$$(\Psi'_+)_{\bar{i}} = (u_a)_{ij}(\Psi_+)_{\bar{j}}, \quad (2.9)$$

We have a similar relation for the antichiral sector.

The third ingredient is the generalized Dirac operator which is an operator in the space \mathcal{H}_M . Since we defined \mathcal{H}_M and \mathcal{A}_M so as to incorporate supersymmetry, let us define the supersymmetric Dirac operator by the following matrix form:

$$\mathcal{D}_M = -i \begin{pmatrix} 0 & \bar{\mathcal{D}}_{i\bar{j}} \\ \mathcal{D}_{\bar{i}j} & 0 \end{pmatrix}, \quad (2.10)$$

where

$$\mathcal{D}_{i\bar{j}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\bar{\sigma}^\mu\partial_\mu & 0 \\ \square & 0 & 0 \end{pmatrix}, \quad \bar{\mathcal{D}}_{i\bar{j}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\sigma^\mu\partial_\mu & 0 \\ \square & 0 & 0 \end{pmatrix}, \quad (2.11)$$

and $\square = \partial^\mu\partial_\mu$.

The operator $\mathcal{D}_{i\bar{j}}$ operates on $(\Psi_+)_{\bar{j}}$ and generates the element in \mathcal{H}_- in the following way,

$$(\Psi'_-)_{\bar{i}} = \frac{1}{m_0}\mathcal{D}_{i\bar{j}}(\Psi_+)_{\bar{j}}, \quad (2.12)$$

$$\begin{cases} \varphi'_-(x) &= F_+(x)/m_0, \\ \psi'^{\dot{\alpha}}_-(x) &= i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_{+\alpha}(x)/m_0, \\ F'_-(x) &= \square\varphi_+(x)/m_0. \end{cases} \quad (2.13)$$

As a matter of fact, we can confirm that the left-hand side of Eq.(2.13) transforms as Eq.(2.4) for the antichiral supermultiplet by applying Eq.(2.2) on the right-hand side. Similarly, for the operator $\bar{\mathcal{D}}_{i\bar{j}}$ we have the following formula:

$$(\Psi'_+)_{\bar{i}} = \frac{1}{m_0}\bar{\mathcal{D}}_{i\bar{j}}(\Psi_-)_{\bar{j}}. \quad (2.14)$$

Let us denote the element in $\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_-$ as

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}, \quad (2.15)$$

then Eq.(2.12) and Eq.(2.14) are put together in the following form:

$$\Psi' = \frac{i}{m_0}\mathcal{D}_M\Psi. \quad (2.16)$$

By applying the supersymmetry transformation, Eq.(2.2) and Eq.(2.4), we can show that

$$[\delta_\xi, \mathcal{D}_M] = 0. \quad (2.17)$$

Thus the operator \mathcal{D}_M given by Eq.(2.10) is invariant under the supersymmetry transformation.

Here a brief comment is in order. If we introduce anti-commuting parameters $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$, an element of the algebra may be expressed by a superfield. For example, Eq.(2.5) corresponds to the following chiral superfield,

$$\Phi_a(x_+) = \varphi_a(x_+) + \sqrt{2}\theta\psi_a(x_+) + \theta\theta F_a(x_+), \quad (2.18)$$

where $x_+^\mu = x^\mu + \theta\sigma^\mu\bar{\theta}$. Then the multiplication rule, Eq.(2.7) is obtained since products of two chiral superfields are again chiral superfields. Note that the set of superfields does not form the Hilbert space since the norm of the superfield is not defined. This is the reason why we constructed \mathcal{H}_M and \mathcal{A}_M using the component fields.

III. THE SPECTRAL TRIPLE IN THE FRAMEWORK OF SUPERSYMMETRY

We first examine that the $(\mathcal{A}_M, \mathcal{H}_M, \mathcal{D}_M)$ introduced in the previous section play the role of the spectral triple of NCG. Since NCG is formulated in the framework of the Euclidean signature, we change the signature from the Minkowskian to the Euclidean one by the Wick rotation.

The functional space \mathcal{H}_M has been defined to include the scalar functions in addition to the spinor functions

in such a way that they form a supermultiplet. In the Euclidean signature, the algebra of $SL(2, C)$ turns out the algebra of $SU(2) \otimes SU(2)$ under the Wick rotation. So, the Weyl spinors $\psi_{+\alpha}$ and $\psi_{-\dot{\alpha}}$ in \mathcal{H}_M are replaced by ρ_α and $\omega^{\dot{\alpha}}$ where they transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of $SU(2) \otimes SU(2)$, respectively. Here $\alpha = 1, 2$ is the index of the first $SU(2)$ and $\dot{\alpha} = 1, 2$ is the index of the second $SU(2)$. The upper index is related to the complex conjugate of the lower index by $\rho^1 = \rho_2^*, \rho^2 = -\rho_1^*$; $\omega^{\dot{1}} = \omega_2^*, \omega^{\dot{2}} = -\omega_1^*$. These functions in \mathcal{H}_M are assumed to be square integrable on the Euclidean compact support so that \mathcal{H}_M is the Hilbert space.

The element of \mathcal{H}_M is denoted by the same form as Eq.(2.15) but now Ψ_+ and Ψ_- are given by.

$$(\Psi_+)_i = (\varphi_+, \rho_\alpha, F_+), \quad (3.1)$$

$$(\Psi_-)_{\bar{i}} = (\varphi_-, \omega^{\dot{\alpha}}, F_-). \quad (3.2)$$

These functions are Euclidean counterparts of Eq.(2.1) and Eq.(2.3). Note that the supersymmetry transformations, Eq.(2.2) and Eq.(2.4) are not defined in the Euclidean NCG since the supersymmetry relates scalar functions to spinor functions which are represented by the Lorentz group, $SL(2, C)$ in the Minkowskian signature.

The element of \mathcal{A}_+ which corresponds to Eq.(2.5) is now given by

$$(u_a)_{ij} = \frac{1}{m_0} \begin{pmatrix} \varphi_a & 0 & 0 \\ \rho_{a\alpha} & \varphi_a & 0 \\ F_a & -\rho_a^{\alpha*} & \varphi_a \end{pmatrix} \in \mathcal{A}_+, \quad (3.3)$$

where $\rho_{a\alpha}$ is $(\frac{1}{2}, 0)$ of $SU(2) \otimes SU(2)$. Similarly, the element of \mathcal{A}_- corresponding to Eq.(2.8) is given by

$$(\bar{u}_a)_{\bar{i}\bar{j}} = \frac{1}{m_0} \begin{pmatrix} \varphi_a^* & 0 & 0 \\ \omega_{a\dot{\alpha}}^* & \varphi_a^* & 0 \\ F_a^* & -\omega_{a\dot{\alpha}}^* & \varphi_a^* \end{pmatrix} \in \mathcal{A}_-, \quad (3.4)$$

and $\omega_{a\dot{\alpha}}^*$ is $(0, \frac{1}{2})$ of $SU(2) \otimes SU(2)$. The multiplication rule of Eq.(2.6) also holds by replacing $\psi_{a\alpha}$ and $\psi_a^{\dot{\alpha}}$ with $\rho_{a\alpha}$ and $\rho_a^{\alpha*}$, respectively. For the algebra \mathcal{A}_- , $\bar{\psi}_a^{\dot{\alpha}}$ and $\psi_{a\dot{\alpha}}$ should be replaced with $\omega^{\dot{\alpha}}$ and ω_a^* , respectively. These algebras \mathcal{A}_\pm are represented in the Hilbert space \mathcal{H}_M by the relation of the same form as Eq.(2.9).

In order to obtain the Dirac operator in the Euclidean signature we replace the Minkowskian signature $g^{\mu\nu}$ by $\eta^{\mu\nu}$. and the Pauli matrices by

$$\sigma_E^\mu = (i\sigma^0, \sigma^i), \quad \bar{\sigma}_E^\mu = (i\sigma^0, -\sigma^i). \quad (3.5)$$

Then from the supersymmetric Dirac operator (2.10) and (2.11) we have

$$\mathcal{D}_M = -i \begin{pmatrix} 0 & \bar{\mathcal{D}}_E \\ \mathcal{D}_E & 0 \end{pmatrix}, \quad (3.6)$$

where \mathcal{D}_E and $\bar{\mathcal{D}}_E$ are given by

$$\mathcal{D}_E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\bar{\sigma}_E^\mu \partial_\mu & 0 \\ \square_E & 0 & 0 \end{pmatrix}, \quad (3.7)$$

and

$$\bar{\mathcal{D}}_E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\sigma_E^\mu \partial_\mu & 0 \\ \square_E & 0 & 0 \end{pmatrix}, \quad (3.8)$$

with

$$\square_E = \eta^{\mu\nu} \partial_\mu \partial_\nu = \partial_0^2 + \partial_i^2. \quad (3.9)$$

Since σ_E^μ and $\bar{\sigma}_E^\mu$ are the $(\frac{1}{2}, \frac{1}{2})$ tensors of $SU(2) \otimes SU(2)$, we have the following relations:

$$\rho'_\alpha = i(\sigma_E^\mu)_{\alpha\dot{\alpha}} \partial_\mu \omega^{\dot{\alpha}}, \quad (3.10)$$

$$\omega'^{\dot{\alpha}} = i(\bar{\sigma}_E^\mu)^{\dot{\alpha}\alpha} \partial_\mu \rho_\alpha. \quad (3.11)$$

Therefore, $i\mathcal{D}_M \Psi, \Psi \in \mathcal{H}_M$ is again the element of \mathcal{H}_M and the same relation as Eq.(2.16) still holds in the Euclidean signature.

Now, the Dirac operator given by Eq.(3.6) has the following property:

- (a) \mathcal{D}_M has the real eigenvalues.

Let us compute the eigenvalues of \mathcal{D}_M for the Euclidean d -dimensional torus, $M = T^d$ with circumference 2π , where d is the space-time dimension, $d = 4$. After a Fourier transform,

$$\Psi(x) = \Psi_n e^{-i \sum n_i x_i} \quad (n_i = 0, \pm 1, \dots), \quad (3.12)$$

the eigenvalue equation reads as follows,

$$\mathcal{D}_M(n) \Psi_n = \lambda_n \Psi_n, \quad (3.13)$$

where

$$\mathcal{D}_M(n) = \begin{pmatrix} 0 & \bar{\mathcal{D}}(n) \\ \mathcal{D}(n) & 0 \end{pmatrix}. \quad (3.14)$$

In Eq.(3.14), $\mathcal{D}(n)$ and $\bar{\mathcal{D}}(n)$ are given by

$$\mathcal{D}(n) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & n_0 + in_3 & in_1 + n_2 & 0 \\ 0 & in_1 - n_2 & n_0 - in_3 & 0 \\ in^2 & 0 & 0 & 0 \end{pmatrix}, \quad (3.15)$$

and

$$\bar{\mathcal{D}}(n) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & n_0 - in_3 & -in_1 - n_2 & 0 \\ 0 & -in_1 + n_2 & n_0 + in_3 & 0 \\ in^2 & 0 & 0 & 0 \end{pmatrix}, \quad (3.16)$$

where $n^2 = n_0^2 + n_1^2 + n_2^2 + n_3^2$. The characteristic equation amounts to

$$\det |\mathcal{D}_M(n) - \lambda \mathbf{1}_8| = (\lambda^2 - n^2)^4 = 0, \quad (3.17)$$

which gives the fourthly degenerate eigenvalues of \mathcal{D}_M as

$$\lambda_n = \pm \sqrt{n_0^2 + n_1^2 + n_2^2 + n_3^2}. \quad (3.18)$$

For large $|\lambda_n|$ there are about $(\pi^2/2)|\lambda_n|^4$ eigenvalues inside the four dimensional ball with the radius $|\lambda_n|$. If we arrange $|\lambda_n|$ in an increasing sequence, we obtain

$$|\lambda_n| \approx \left(\frac{2n}{\pi^2}\right)^{1/4}, \quad (3.19)$$

for large n .

- (b) The resolvent $R(\lambda; \mathcal{D}_M) = (\mathcal{D}_M - \lambda \mathbf{1}_8)^{-1}$, $\lambda \notin \mathbb{R}$ of \mathcal{D}_M is compact.

As a matter of fact, for any $\varepsilon > 0$ with sufficiently large N the norm of the resolvent obeys the following relation:

$$\|R(\lambda; \mathcal{D}_M)\| < |\lambda_{N+1} - \lambda|^{-1} < \varepsilon, \quad (3.20)$$

on the orthogonal of a N dimensional subspace of \mathcal{H}_M .

- (c) The commutators, $[\mathcal{D}_M, a] = \mathcal{D}_M a - a \mathcal{D}_M$ are bounded for any $a \in \mathcal{A}_M$.

The relations due to the supersymmetry such as Eq.(2.9) and Eq.(2.14) in the Minkowskian signature are transferred to the corresponding relations in the Euclidean signature. This implies that if Ψ is an element of \mathcal{H}_M , then $a\Psi$ and $\mathcal{D}_M\Psi$ are also the element of \mathcal{H}_M , so that $[\mathcal{D}_M, a]$ is a bounded operator which maps a finite element of the Hilbert space to another finite element.

A $\mathbb{Z}/2$ grading of the Hilbert space \mathcal{H}_M is given by an operator γ_M in \mathcal{H}_M which is defined by

$$\gamma_M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.21)$$

on the basis such that $\gamma_M(\Psi_+) = -1$ and $\gamma_M(\Psi_-) = 1$. Obviously γ_M satisfies $\gamma_M^* = \gamma_M$, $\gamma_M^2 = 1$ and obeys the following relations:

$$\gamma_M a = a \gamma_M, \quad \forall a \in \mathcal{A}_M, \quad (3.22)$$

$$\mathcal{D}_M \gamma_M = -\gamma_M \mathcal{D}_M. \quad (3.23)$$

In the rest of this section, let us consider the real structure of the Hilbert space. First let us define the charge conjugation. For the state $\Psi \in \mathcal{H}_M$ the charge conjugate state Ψ^c is given by

$$\Psi^c = \begin{pmatrix} \Psi_+^c \\ \Psi_-^c \end{pmatrix}, \quad (3.24)$$

and

$$(\Psi_+^c)_i = (\varphi_+^*, \rho^*, F_+^*), \quad (3.25)$$

$$(\Psi_-^c)_{\bar{i}} = (\varphi_-^*, \omega_{\dot{\alpha}}, F_-^*). \quad (3.26)$$

Then let us define the antilinear operator \mathcal{J}_M by

$$\Psi^c = \mathcal{J}_M \Psi = C \Psi^*, \quad (3.27)$$

so that it is given by

$$\mathcal{J}_M = C \otimes *, \quad (3.28)$$

where C is the following charge conjugation matrix;

$$C = \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array} \right), \quad (3.29)$$

and $*$ is the complex conjugation (hermitian conjugation for matrices). The operator \mathcal{J}_M obeys the following relation:

$$\mathcal{J}_M \gamma_M = \gamma_M \mathcal{J}_M. \quad (3.30)$$

The real structure \mathcal{J}_M is now expressed for the basis of the Hilbert space given by $(\Psi, \Psi^c)^T$ in the following matrix form:

$$\mathcal{J}_M = \begin{pmatrix} 0 & \mathcal{J}_M^{-1} \\ \mathcal{J}_M & 0 \end{pmatrix}. \quad (3.31)$$

On the same basis, the Dirac operator D_M and the $\mathbb{Z}/2$ grading Γ_M is expressed by

$$D_M = \begin{pmatrix} \mathcal{D}_M & 0 \\ 0 & \mathcal{J}_M \mathcal{D}_M \mathcal{J}_M^{-1} \end{pmatrix}, \quad (3.32)$$

and

$$\Gamma_M = \begin{pmatrix} \gamma_M & 0 \\ 0 & \gamma_M \end{pmatrix}. \quad (3.33)$$

The real structure defined by Eq.(3.31) satisfies the following relations:

$$\mathcal{J}_M^2 = 1, \quad (3.34)$$

$$\mathcal{J}_M D_M = D_M \mathcal{J}_M, \quad (3.35)$$

$$\mathcal{J}_M \Gamma_M = \Gamma_M \mathcal{J}_M. \quad (3.36)$$

In the Minkowskian signature, we shall define \mathcal{J}_M by the same relation as Eq.(3.27). In this case, however, the charge conjugation is defined for Dirac spinors. A Dirac spinor ψ is composed of two Weyl spinors, χ and ξ as

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}. \quad (3.37)$$

Then, the state Ψ and its charge conjugate state Ψ^c in \mathcal{H}_M is denoted by

$$\Psi = \left(\varphi_\chi, \chi_\alpha, F_\chi, \varphi_\xi^*, \bar{\xi}^{\dot{\alpha}}, F_\xi^* \right)^T, \quad (3.38)$$

and

$$\Psi^c = (\varphi_\xi, \xi_\alpha, F_\xi, \varphi_\chi^*, \bar{\chi}^{\dot{\alpha}}, F_\chi^*)^T. \quad (3.39)$$

The charge conjugation matrix in Eq.(3.28) is now given by

$$C = \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & \mathbf{0} & & 0 & \varepsilon_{\alpha\beta} & 0 \\ & & & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & & \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \end{array} \right). \quad (3.40)$$

The $\mathbb{Z}/2$ grading in the Minkowskian signature is defined by

$$\gamma_M = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad (3.41)$$

since the state Ψ and its charge conjugate state Ψ^c in \mathcal{H}_M are given by Eq.(3.38) and Eq.(3.39) so that

$$\gamma_M(\Psi_+) = \gamma_M(\Psi_-^*) = \gamma_M(\Psi_+^c) = -i, \quad (3.42)$$

$$\gamma_M(\Psi_-) = \gamma_M(\Psi_+^*) = \gamma_M(\Psi_-^c) = i. \quad (3.43)$$

Although γ_M and C are anticommuting, \mathcal{J}_M defined by Eq.(3.28) commutes with γ_M and satisfies the same relation as Eq.(3.30). As a result, the real structure J_M as well as D_M and Γ_M in the Minkowskian signature satisfy the same relation as Eq.(3.34)–Eq.(3.36).

IV. INTERNAL FLUCTUATIONS AND VECTOR SUPERMULTIPLY

In the nonsupersymmetric noncommutative geometry the vector field was introduced as the internal fluctuation on the metric by the following formula,

$$A = \sum a_i [D, b_i] = i\gamma^\mu \sum a_i \partial_\mu b_i, \quad (4.1)$$

$$A_\mu = i \sum a_i \partial_\mu b_i, \quad (4.2)$$

here $a_i, b_i \in \mathcal{A}$ and the Dirac operator D is given by $D = i\gamma^\mu \partial_\mu$ [21, 23].

In order to introduce vector fields in the supersymmetric theory we need two sets of the elements of \mathcal{A}_+ and \mathcal{A}_- ,

$$\Pi_+ = \{u_a; a = 1, 2, \dots, n\} \subset \mathcal{A}_+, \quad (4.3)$$

$$\Pi_- = \{\bar{u}_a; a = 1, 2, \dots, n\} \subset \mathcal{A}_-, \quad (4.4)$$

where u_a and \bar{u}_a are given in the matrix form of Eq.(2.5) and Eq.(2.8). Since the product of chiral (antichiral) supermultiplets is again the chiral (antichiral) supermultiplet, the elements of Π_+ (Π_-) are chosen such the products of two or more u'_a (\bar{u}'_a) do not belong to Π_+ (Π_-) any more. Since we consider supersymmetry, we work

in the Minkowskian signature so that elements of the algebra obey the supersymmetry transformations given by Eq.(2.2) and Eq.(2.4).

We shall define the following scalar, spinor and vector fields as the bilinear form of the two component functions in $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$;

$$m_0^2 C = \sum_a c_a \varphi_a^* \varphi_a, \quad (4.5)$$

$$m_0^2 \chi_\alpha = -i\sqrt{2} \sum_a c_a \varphi_a^* \psi_{a\alpha}, \quad (4.6)$$

$$m_0^2 (M + iN) = -2i \sum_a c_a \varphi_a^* F_a, \quad (4.7)$$

$$m_0^2 A_\mu = -i \sum_a c_a [(\varphi_a^* \partial_\mu \varphi_a - \partial_\mu \varphi_a^* \varphi_a) - i\bar{\psi}_{a\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \psi_{a\alpha}], \quad (4.8)$$

$$m_0^2 \lambda_\alpha = \sqrt{2}i \sum_a c_a \left(F_a^* \psi_{a\alpha} - i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\psi}_a^{\dot{\alpha}} \partial_\mu \varphi_a \right), \quad (4.9)$$

$$m_0^2 D = \sum_a c_a [2F_a^* F_a - 2(\partial^\mu \varphi_a^* \partial_\mu \varphi_a) + i\{\partial_\mu \bar{\psi}_{a\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \psi_{a\alpha} - \bar{\psi}_{a\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \partial_\mu \psi_{a\alpha}\}], \quad (4.10)$$

where c_a are the real coefficients. Using Eq.(2.2) and Eq.(2.4), we can show that these fields have the following transformation property of the vector supermultiplet expressed by

$$\delta_\xi C = i\xi^\alpha \chi_\alpha - i\bar{\xi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}, \quad (4.11)$$

$$\delta_\xi \chi_\alpha = -i\sigma_{\alpha\dot{\alpha}}^\mu \bar{\xi}^{\dot{\alpha}} (-A_\mu + i\partial_\mu C) + \xi_\alpha (M + iN), \quad (4.12)$$

$$\frac{1}{2}\delta_\xi (M + iN) = \bar{\xi}_{\dot{\alpha}} \left(\bar{\lambda}^{\dot{\alpha}} + i\bar{\sigma}^{\dot{\alpha}\alpha}_\mu \partial_\mu \chi_\alpha \right), \quad (4.13)$$

$$\delta_\xi A^\mu = i\xi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\lambda}^{\dot{\alpha}} + i\bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \lambda_\alpha + \xi^\alpha \partial^\mu \chi_\alpha + \bar{\xi}_{\dot{\alpha}} \partial^\mu \bar{\chi}^{\dot{\alpha}}, \quad (4.14)$$

$$\delta_\xi \lambda_\alpha = \sigma_{\alpha}^{\mu\nu\beta} \xi_\beta (\partial_\mu A_\nu - \partial_\nu A_\mu) + i\xi_\alpha D, \quad (4.15)$$

$$\delta_\xi D = -\xi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{\lambda}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \bar{\sigma}^{\dot{\alpha}\alpha}_\mu \partial_\mu \lambda_\alpha. \quad (4.16)$$

If we express these fields as the superfield, we have

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C + \theta^\alpha (i\chi_\alpha) + \bar{\theta}_{\dot{\alpha}} (-i\bar{\chi}^{\dot{\alpha}}) + \theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} (-A_\mu) \\ & + \theta\bar{\theta} \left[\frac{i}{2}(M + iN) \right] + \bar{\theta}\bar{\theta} \left[-\frac{i}{2}(M - iN) \right] \\ & + \theta\bar{\theta}\bar{\theta}_{\dot{\alpha}} \left[i \left(\bar{\lambda}^{\dot{\alpha}} + \frac{i}{2}\bar{\sigma}^{\dot{\alpha}\alpha}_\mu \partial_\mu \chi_\alpha \right) \right] \\ & + \bar{\theta}\bar{\theta}\theta^\alpha \left[-i \left(\lambda_\alpha + \frac{i}{2}\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \bar{\chi}^{\dot{\alpha}} \right) \right] \\ & + \theta\bar{\theta}\bar{\theta} \left(\frac{1}{2}D + \frac{1}{4}\square C \right). \end{aligned} \quad (4.17)$$

When we define the vector supermultiplet given by Eq.(4.5)–Eq.(4.10), there is an ambiguity due to the choice of the algebraic elements. In order to see this, we consider two arbitrary elements of the algebra given

by $u_0 \in \mathcal{A}_+$ and $\bar{u}_0 \in \mathcal{A}_-$. It turns out that the following functions obtained by these elements obey the supersymmetry transformation of the vector supermultiplet given by Eq.(4.11)–Eq.(4.16),

$$m_0 C_0 = \varphi_0 + \varphi_0^*, \quad (4.18)$$

$$m_0 \chi_{0\alpha} = -i\sqrt{2}\psi_{0\alpha}, \quad (4.19)$$

$$m_0 (M_0 + iN_0) = -2iF_0, \quad (4.20)$$

$$m_0 A_{0\mu} = -i\partial_\mu(\varphi_0 - \varphi_0^*), \quad (4.21)$$

$$\lambda_{0\alpha} = 0, \quad (4.22)$$

$$D_0 = 0. \quad (4.23)$$

Then we can redefine C, χ_α, M, N such that

$$C \rightarrow C + C_0 = 0, \quad (4.24)$$

$$\chi_\alpha \rightarrow \chi_\alpha + \chi_{0\alpha} = 0, \quad (4.25)$$

$$M + iN \rightarrow (M + M_0) + i(N + N_0) = 0. \quad (4.26)$$

To choose C, χ_α, M and N in the vector supermultiplet to be zero is called the Wess-Zumino gauge. This gauge is realized in Eq.(4.5)–Eq.(4.10) by the following condition:

$$\begin{cases} \sum_a c_a \varphi_a^* \varphi_a = 0, \\ \sum_a c_a \varphi_a^* \psi_a^\alpha = 0, \\ \sum_a c_a \varphi_a^* F_a = 0. \end{cases} \quad (4.27)$$

Hereafter let us call Eq.(4.27) the Wess-Zumino gauge condition.

Now let us calculate the internal fluctuation of the supersymmetric Dirac operator \mathcal{D}_M given by Eq.(2.10). The modified Dirac operator is denoted by

$$\tilde{\mathcal{D}}_M = -i \begin{pmatrix} 0 & \tilde{\mathcal{D}}_{i\bar{j}} \\ \tilde{\mathcal{D}}_{\bar{i}j} & 0 \end{pmatrix}. \quad (4.28)$$

First, we consider the fluctuation due to $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$. The contribution to $\tilde{\mathcal{D}}_{i\bar{j}}$ is given by the following matrix form:

$$\begin{aligned} V_{i\bar{j}} &= -2 \sum_a c_a (\bar{u}_a)_{\bar{i}\bar{k}} [i\mathcal{D}_M, u_a]_{\bar{k}j} \\ &= -2 \sum_a c_a (\bar{u}_a)_{\bar{i}\bar{k}} \mathcal{D}_{\bar{k}\ell}(u_a)_{\ell j}, \end{aligned} \quad (4.29)$$

and the contribution to $\tilde{\mathcal{D}}_{\bar{i}j}$ is given by

$$\begin{aligned} \bar{V}_{i\bar{j}} &= 2 \sum_a c_a (u_a)_{ik} [i\mathcal{D}_M, \bar{u}_a]_{k\bar{j}} \\ &= 2 \sum_a c_a (u_a)_{ik} \bar{\mathcal{D}}_{k\bar{\ell}}(\bar{u}_a)_{\bar{\ell}\bar{j}}. \end{aligned} \quad (4.30)$$

We shall calculate in the Wess-Zumino gauge. Using the definition of the vector supermultiplet given by Eq.(4.5)–Eq.(4.13), we obtain the following result:

$$V_{i\bar{j}} = - \begin{pmatrix} 0 & 0 & 0 \\ i\sqrt{2}\lambda^\alpha & -\bar{\sigma}^{\mu\dot{\alpha}\alpha} A_\mu & 0 \\ D + i\partial^\mu A_\mu + 2iA_\mu \partial^\mu & i\sqrt{2}\lambda^\alpha & 0 \end{pmatrix}, \quad (4.31)$$

and

$$\bar{V}_{i\bar{j}} = \begin{pmatrix} 0 & 0 & 0 \\ -i\sqrt{2}\lambda_\alpha & \sigma_{\alpha\dot{\alpha}}^\mu A_\mu & 0 \\ D - i\partial^\mu A_\mu - 2iA_\mu \partial^\mu & -i\sqrt{2}\lambda_\alpha & 0 \end{pmatrix}. \quad (4.32)$$

Next, let us consider the additional fluctuation due to $u_{ab} = u_a u_b \in \mathcal{A}_+$ and $\bar{u}_{ab} = \bar{u}_a \bar{u}_b \in \mathcal{A}_-$, where $a, b = 1, \dots, n$. This fluctuation is not contained in the fluctuation due to $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$ since $u_{ab} \notin \Pi_+$ and $\bar{u}_{ab} \notin \Pi_-$. The component fields of u_{ab} are expressed by the matrix form of Eq.(2.5) and each field is given by

$$u_{ab} = \{\varphi_{ab}, \psi_{ab\alpha}, F_{ab}\}, \quad (4.33)$$

where

$$\varphi_{ab} = \frac{1}{m_0} \varphi_a \varphi_b, \quad (4.34)$$

$$\psi_{ab\alpha} = \frac{1}{m_0} (\psi_{a\alpha} \varphi_b + \varphi_a \psi_{b\alpha}), \quad (4.35)$$

$$F_{ab} = \frac{1}{m_0} (\varphi_a F_b + F_a \varphi_b - \psi_a^\alpha \psi_{b\alpha}). \quad (4.36)$$

The component fields of \bar{u}_{ab} are the complex conjugate functions of Eq.(4.34)–Eq.(4.36).

It turns out that the gauge-covariant form of $\tilde{\mathcal{D}}_M$ is obtained by considering the following fluctuation due to u_{ab} and \bar{u}_{ab} :

$$\begin{aligned} V'_{i\bar{j}} &= 2 \sum c_a c_b (\bar{u}_{ab})_{\bar{i}\bar{k}} [i\mathcal{D}_M, u_{ab}]_{\bar{k}j} \\ &= 2 \sum c_a c_b (\bar{u}_{ab})_{\bar{i}\bar{k}} \mathcal{D}_{\bar{k}\ell}(u_{ab})_{\ell j}, \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} \bar{V}'_{i\bar{j}} &= 2 \sum c_a c_b (u_{ab})_{ik} [i\mathcal{D}_M, \bar{u}_{ab}]_{k\bar{j}} \\ &= 2 \sum c_a c_b (u_{ab})_{ik} \bar{\mathcal{D}}_{k\bar{\ell}}(\bar{u}_{ab})_{\bar{\ell}\bar{j}}. \end{aligned} \quad (4.38)$$

Taking into account the Wess-Zumino gauge condition given by Eq.(4.27), we obtain

$$V'_{31} = \bar{V}'_{31} = -A_\mu A^\mu, \quad (4.39)$$

and other matrix elements turn out to be zero. The fluctuation due to higher order products of u_a or \bar{u}_a such as $u_{abc} = u_a u_b u_c$ or $\bar{u}_{abc} = \bar{u}_a \bar{u}_b \bar{u}_c$ vanishes due to the Wess-Zumino gauge condition. Thus the total fluctuation in the Wess-Zumino gauge amounts to

$$V_{i\bar{j}}^{WZ} = V_{i\bar{j}} + V'_{i\bar{j}}, \quad (4.40)$$

$$\bar{V}_{i\bar{j}}^{WZ} = \bar{V}_{i\bar{j}} + \bar{V}'_{i\bar{j}}, \quad (4.41)$$

and the Dirac operator with fluctuation denoted by Eq.(4.28) is finally given by

$$\begin{aligned} \tilde{\mathcal{D}}_{i\bar{j}} &= \mathcal{D}_{i\bar{j}} + V_{i\bar{j}}^{WZ} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2}\lambda^\alpha & i\bar{\sigma}^\mu D_\mu & 0 \\ \mathcal{D}_\mu D^\mu - D & -i\sqrt{2}\lambda^\alpha & 0 \end{pmatrix}, \end{aligned} \quad (4.42)$$

and

$$\begin{aligned}\widetilde{\mathcal{D}}_{i\bar{j}} &= \overline{\mathcal{D}}_{i\bar{j}} + \overline{V}_{i\bar{j}}^{WZ} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2}\lambda_\alpha & i\sigma^\mu \mathcal{D}_\mu & 0 \\ \mathcal{D}_\mu \mathcal{D}^\mu + D & -i\sqrt{2}\lambda_{\dot{\alpha}} & 0 \end{pmatrix},\end{aligned}\quad (4.43)$$

where \mathcal{D}_μ is the covariant derivative,

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu. \quad (4.44)$$

V. INTERNAL DEGREES OF FREEDOM AND FINITE GEOMETRY

Internal degrees of freedom are introduced in NCG by the finite geometry described by the spectral triple $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$. Let us consider in this section the color degrees of freedom of quarks and let $\mathcal{A}_F = M_3(\mathbb{C})$, the algebra of 3×3 complex matrices.

We let \mathcal{H}_F be the Hilbert space with basis of the labels q_L^a and q_R^a of quark-supermultiplets which consist of quarks, squarks and auxiliary fields. Here a is the color index, $a = 1, 2, 3$ and L, R denote the eigenstates of the $\mathbb{Z}/2$ -grading γ_F , which is defined by

$$\gamma_F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.1)$$

For the basis of \mathcal{H}_F given by

$$Q^a = \begin{pmatrix} q_L^a \\ q_R^a \end{pmatrix} \in \mathcal{H}_F, \quad (5.2)$$

we have $\gamma_F(q_L^a) = -1$, and $\gamma_F(q_R^a) = 1$. We shall define the antiquark-supermultiplet states in \mathcal{H}_F as follows;

$$\begin{aligned}(q_a^c)_L &= (q_R^a)^*, \\ (q_a^c)_R &= (q_L^a)^*,\end{aligned}\quad (5.3)$$

and

$$Q_a^c = \begin{pmatrix} (q_a^c)_L \\ (q_a^c)_R \end{pmatrix} \in \mathcal{H}_F. \quad (5.4)$$

Let us define the antilinear operator \mathcal{J}_F by

$$\mathcal{J}_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes *, \quad (5.5)$$

where $*$ is the complex conjugation. Then the antiquark-supermultiplet Q_a^c is related to Q_a by

$$Q_a^c = \mathcal{J}_F Q^a. \quad (5.6)$$

The Dirac operator D_F on the basis $(Q^a, Q_a^c)^T$ is given by

$$D_F = \begin{pmatrix} \mathcal{D}_F & 0 \\ 0 & \mathcal{J}_F \mathcal{D}_F \mathcal{J}_F^{-1} \end{pmatrix}. \quad (5.7)$$

Here \mathcal{D}_F is defined by

$$\mathcal{D}_F = \begin{pmatrix} 0 & m^T \\ m & 0 \end{pmatrix}, \quad (5.8)$$

and m is the mass matrix with respect to the family index of quark-supermultiplets.

On the same basis, the $\mathbb{Z}/2$ -grading Γ_F and the real structure J_F are expressed as follows:

$$\Gamma_F = \begin{pmatrix} \gamma_F & 0 \\ 0 & \gamma_F \end{pmatrix}, \quad (5.9)$$

and

$$J_F = \begin{pmatrix} 0 & \mathcal{J}_F^{-1} \\ \mathcal{J}_F & 0 \end{pmatrix}. \quad (5.10)$$

Since γ_F and \mathcal{D}_F are anticommuting we have

$$D_F \Gamma_F = -\Gamma_F D_F. \quad (5.11)$$

Furthermore, J_F, D_F , and Γ_F obey the following relations;

$$J_F^2 = 1, \quad (5.12)$$

$$J_F D_F = D_F J_F, \quad (5.13)$$

$$J_F \Gamma_F = -\Gamma_F J_F. \quad (5.14)$$

These relations imply that the K-theoretic dimension of the finite space is 6. This dimension is required to avoid the fermion doubling problem [13, 24].

The spectral triple (A_0, H_0, D_0) for the product of the space-time manifold M by the finite geometry F is given by

$$A_0 = \mathcal{A}_M \otimes \mathcal{A}_F, \quad (5.15)$$

$$H_0 = \mathcal{H}_M \otimes \mathcal{H}_F, \quad (5.16)$$

$$D_0 = D_M \otimes 1 + \Gamma_M \otimes D_F. \quad (5.17)$$

The wave functions of the quark-supermultiplets in \mathcal{H} are in $(\Psi_+, \Psi_-) \otimes (q_L^a, q_R^a)$. In order to evade fermion doubling [25, 26] we impose that the physical quark wave functions obey the following condition:

$$\gamma = \gamma_M \gamma_F = 1. \quad (5.18)$$

Then for the left-handed quark-supermultiplet we have

$$\begin{aligned}\Psi_L^a(x) &= q_L^a \otimes (\Psi_+(x))_i \\ &= q_L^a \otimes (\varphi_+(x), \psi_{+\alpha}(x), F_+(x)),\end{aligned}\quad (5.19)$$

in the Minkowskian signature and the wave functions of quark, squark, and auxiliary field amount to

$$q_{L\alpha}^a(x) = q_L^a \otimes \psi_{+\alpha}(x), \quad (5.20)$$

$$\tilde{q}_L^a(x) = q_L^a \otimes \varphi_+(x), \quad (5.21)$$

$$F_L^a(x) = q_L^a \otimes F_+(x). \quad (5.22)$$

For the right-handed quark-supermultiplet we have

$$\Psi_R^a(x) = q_R^a \otimes (\Psi_-(x))_{\bar{i}}, \quad (5.23)$$

and the wave functions of the component fields are given by

$$q_R^{a\dot{\alpha}}(x) = q_R^a \otimes \psi_{\dot{\alpha}}^-(x), \quad (5.24)$$

$$\tilde{q}_R^a(x) = q_R^a \otimes \varphi_-(x), \quad (5.25)$$

$$F_R^a(x) = q_R^a \otimes F_-(x). \quad (5.26)$$

The elements of the algebra A_0 given by Eq.(5.15) are the matrix-valued supermultiplet, i.e., the component functions of u_a and \bar{u}_a in Eq.(4.3) and Eq.(4.4), respectively, are 3×3 complex matrix functions. Then the functions in the vector supermultiplet defined by Eq.(4.5)–Eq.(4.10) are also 3×3 complex matrix functions. In the Wess-Zumino gauge we consider A_μ, λ_α and D . Among them, A_μ and D are hermitian so that they are parametrized by

$$A_\mu(x) = \sum_{\ell=0, \dots, 8} A_\mu^\ell(x) \frac{t_\ell}{2}, \quad (5.27)$$

$$D(x) = \sum_{\ell=0, \dots, 8} D^\ell(x) \frac{t_\ell}{2}, \quad (5.28)$$

where $A_\mu^\ell(x)$ and $D^\ell(x)$ are real functions and t_ℓ are 3×3 Gell-Mann matrices. The complex matrix function λ_α is expressed by

$$\lambda_\alpha(x) = \sum_{\ell=0, \dots, 8} \lambda_\alpha^\ell(x) \frac{t_\ell}{2}, \quad (5.29)$$

and $\lambda_\alpha^\ell(x)$ are complex functions.

Since the vector supermultiplet originates from the fluctuation of the metric, the matrix-valued functions A_μ, λ_α and D should be traceless because the trace part does not affect the metric [21]. This amounts to remove t_0 from the summation in Eq.(5.27)–Eq.(5.29). As a result these functions have the same form as the adjoint representation of $SU(3)$. In the next section we shall show that the vector supermultiplet actually satisfies the super Yang-Mills action with the $SU(3)$ gauge symmetry.

VI. SUPERSYMMETRIC QCD AND THE SPECTRAL ACTION PRINCIPLE

In the Euclidean NCG models without supersymmetry, the bosonic part of the action is obtained by the spectral action principle, which asserts that the action depends only on the spectrum of the squared Dirac operator. In our noncommutative geometric approach to supersymmetry, we have derived the supersymmetric Dirac operator $\tilde{\mathcal{D}}_M$ given by Eq.(4.28) with Eq.(4.42) and Eq.(4.43).

Let us show that the supersymmetric action for the vector supermultiplet will be obtained by the spectral

action in the heat kernel expansion of the elliptic operator P :

$$\text{Tr}_{L^2} f(P) \simeq \sum_{n \geq 0} c_n a_n(P), \quad (6.1)$$

where $f(x)$ is an auxiliary smooth function on a smooth compact Riemannian manifold M without boundary of dimension 4 [27]. Since the contribution to P from the antiparticles is the same as that of the particles, we consider only the contribution from the particles. Then the elliptic operator P in our case is given by the square of the Wick rotated Euclidean Dirac operator $\tilde{\mathcal{D}}_0$,

$$\tilde{\mathcal{D}}_0 = \tilde{\mathcal{D}}_M + \gamma_M \otimes \mathcal{D}_F, \quad (6.2)$$

where $\tilde{\mathcal{D}}_M$ is obtained from Eq.(4.28), Eq.(4.42) and Eq.(4.43) by a replacement of $g^{\mu\nu} \rightarrow \eta^{\mu\nu}, \sigma^\mu \rightarrow \sigma_E^\mu, \bar{\sigma}^\mu \rightarrow \bar{\sigma}_E^\mu$. Note that the internal fluctuation on \mathcal{D}_F is absent in the QCD model.

The elliptic operator P is expanded into the following form:

$$P = -(\eta^{\mu\nu} \partial_\mu \partial_\nu + \mathbb{A}^\mu \partial_\mu + \mathbb{B}). \quad (6.3)$$

The heat kernel coefficients a_n in Eq.(6.1) are found in [27]. They vanish for n odd, and the first three a_n 's for n even in the flat space are given by

$$a_0(P) = \frac{1}{16\pi^2} \int_M dx^4 \text{tr}_V(\mathbb{I}), \quad (6.4)$$

$$a_2(P) = \frac{1}{16\pi^2} \int_M dx^4 \text{tr}_V(\mathbb{E}), \quad (6.5)$$

$$a_4(P) = \frac{1}{32\pi^2} \int_M dx^4 \times \text{tr}_V \left(\mathbb{E}^2 + \frac{1}{3} \mathbb{E}_{;\mu}{}^\mu + \frac{1}{6} \Omega_{\mu\nu} \Omega^{\mu\nu} \right), \quad (6.6)$$

where \mathbb{E} and the bundle curvature $\Omega^{\mu\nu}$ in the flat space are defined as follows;

$$\mathbb{E} = \mathbb{B} - (\partial_\mu \omega^\mu + \omega_\mu \omega^\mu), \quad (6.7)$$

$$\Omega^{\mu\nu} = \partial^\mu \omega^\nu - \partial^\nu \omega^\mu + [\omega^\mu, \omega^\nu], \quad (6.8)$$

$$\omega^\mu = \frac{1}{2} \mathbb{A}^\mu. \quad (6.9)$$

The coefficients c_n in Eq.(6.1) depend on the functional form of $f(x)$. If $f(x)$ is flat near 0, it turns out that $c_{2k} = 0$ for $k \geq 3$ and the heat kernel expansion (6.1) terminates at $n = 4$ [20].

In Eq.(6.4)–Eq.(6.6), tr_V denotes the trace over the vector bundle V . As for the supersymmetric theory we consider here, sections of the vector bundle V are smooth functions bearing indices which correspond to internal and spin degrees of freedom of the chiral or antichiral supermultiplets. For the spin degrees of freedom, tr_V is the supertrace defined by

$$\begin{aligned} \text{Str } O &= \sum_i \langle i | (-1)^{2s} O | i \rangle \\ &= \sum_b \langle b | O | b \rangle - \sum_f \langle f | O | f \rangle, \end{aligned} \quad (6.10)$$

where s is the spin angular momentum and the states $|b\rangle$ and $|f\rangle$ stand for bosonic and fermionic states, respectively.

Now let us calculate the spectral action for $\tilde{\mathcal{D}}_0^2$ where $\tilde{\mathcal{D}}_0$ is given by Eq.(6.2). In the contribution to the spectral action from $\tilde{\mathcal{D}}_0^2$, the terms including \mathcal{D}_F vanish since $\tilde{\mathcal{D}}_M$ anticommutes with γ_M and

$$\text{Str } \gamma_M^2 = \text{Str } 1 = 0. \quad (6.11)$$

Thus, we consider the following elliptic operator P in the Euclidean signature,

$$P = \tilde{\mathcal{D}}_M^2 = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}. \quad (6.12)$$

By making use of the Wick-rotated expression of Eq.(4.42) and Eq.(4.43), P_{\pm} amounts to

$$\begin{aligned} P_+ &= -\tilde{\mathcal{D}}_E \tilde{\mathcal{D}}_E \\ &= - \begin{pmatrix} \mathcal{D}_\mu \mathcal{D}^\mu - D & -i\sqrt{2}\lambda^\beta & 0 \\ \sqrt{2}\sigma_{E\alpha\dot{\alpha}}^\mu (\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}}) + \sqrt{2}\sigma_{E\alpha\dot{\alpha}}^\mu \bar{\lambda}^{\dot{\alpha}} \mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu \delta_\alpha^\beta + i\sigma_{E\alpha}^{\mu\nu\beta} F_{\mu\nu} & -i\sqrt{2}\lambda_\alpha \\ -2\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} & \sqrt{2}\bar{\lambda}_{\dot{\alpha}} \sigma_{E\alpha}^{\mu\dot{\alpha}\beta} \mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu + D \end{pmatrix}, \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} P_- &= -\tilde{\mathcal{D}}_E \tilde{\mathcal{D}}_E \\ &= - \begin{pmatrix} \mathcal{D}_\mu \mathcal{D}^\mu + D & -\sqrt{2}i\bar{\lambda}_{\dot{\beta}} & 0 \\ \sqrt{2}\bar{\sigma}_E^{\mu\dot{\alpha}\alpha} (\mathcal{D}_\mu \lambda_\alpha) + \sqrt{2}\bar{\sigma}_E^{\mu\dot{\alpha}\alpha} \lambda_\alpha \mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu \delta_{\dot{\alpha}}^{\dot{\beta}} + i\bar{\sigma}_E^{\mu\nu\dot{\alpha}\dot{\beta}} F_{\mu\nu} & -\sqrt{2}i\bar{\lambda}^{\dot{\alpha}} \\ -2\lambda^\alpha \lambda_\alpha & \sqrt{2}\lambda^\alpha \sigma_{E\alpha\dot{\beta}}^\mu \mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu - D \end{pmatrix}. \end{aligned} \quad (6.14)$$

In Eq.(6.13) and Eq.(6.14), $\sigma_E^{\mu\nu}$ and $\bar{\sigma}_E^{\mu\nu}$ are defined by

$$\sigma_E^{\mu\nu} = (i\sigma^{0j}, \sigma^{ij}), \quad (6.15)$$

$$\bar{\sigma}_E^{\mu\nu} = (i\bar{\sigma}^{0j}, \bar{\sigma}^{ij}), \quad (6.16)$$

and

$$\sigma_{\alpha}^{\mu\nu\beta} = \frac{1}{4} (\sigma_{\alpha\dot{\alpha}}^\mu \bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma_{\alpha\dot{\alpha}}^\nu \bar{\sigma}^{\mu\dot{\alpha}\beta}), \quad (6.17)$$

$$\bar{\sigma}^{\mu\nu\dot{\alpha}}_{\dot{\beta}} = \frac{1}{4} (\bar{\sigma}^{\mu\dot{\alpha}\beta} \sigma_{\beta\dot{\beta}}^\nu - \bar{\sigma}^{\nu\dot{\alpha}\beta} \sigma_{\beta\dot{\beta}}^\mu). \quad (6.18)$$

The fields A_μ, λ_α, D are the 3×3 matrices as shown

by Eq.(5.27)–Eq.(5.29). They turn out to be the gauge, gaugino and auxiliary fields. The covariant derivative on spinors, say, λ_α is given by

$$\mathcal{D}_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha - i[A_\mu, \lambda_\alpha], \quad (6.19)$$

and $F_{\mu\nu}$ is the field strength defined by

$$\begin{aligned} F_{\mu\nu} &= i[\mathcal{D}_\mu, \mathcal{D}_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \end{aligned} \quad (6.20)$$

We expand P_{\pm} in the form given by Eq.(6.3). Using the formulae given by Eq.(6.6) and Eq.(6.8), we obtain the following expressions;

$$\mathbb{E}_+ = \mathbb{B}_+ - (\partial_\mu \omega_+^\mu + \omega_{+\mu} \omega_+^\mu) = \begin{pmatrix} -D & -i\sqrt{2}\lambda^\beta & 0 \\ \frac{1}{\sqrt{2}}\sigma_{E\alpha\dot{\alpha}}^\mu (\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}}) & i\sigma_{E\alpha}^{\mu\nu\beta} F_{\mu\nu} & -i\sqrt{2}\lambda_\alpha \\ -2\bar{\lambda}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} & -\frac{1}{\sqrt{2}}(\mathcal{D}_\mu \bar{\lambda}_{\dot{\alpha}}) \bar{\sigma}_E^{\mu\dot{\alpha}\beta} & D \end{pmatrix}, \quad (6.21)$$

and

$$\mathbb{E}_- = \mathbb{B}_- - (\partial_\mu \omega_-^\mu + \omega_{-\mu} \omega_-^\mu) = \begin{pmatrix} D & -i\sqrt{2}\bar{\lambda}_{\dot{\beta}} & 0 \\ \frac{1}{\sqrt{2}}\bar{\sigma}_E^{\mu\dot{\alpha}\alpha} (\mathcal{D}_\mu \lambda_\alpha) & i\bar{\sigma}_E^{\mu\nu\dot{\alpha}\dot{\beta}} F_{\mu\nu} & -i\sqrt{2}\bar{\lambda}^{\dot{\alpha}} \\ -2\lambda^\alpha \lambda_\alpha & -\frac{1}{\sqrt{2}}(\mathcal{D}_\mu \lambda^\alpha) \sigma_{E\alpha\dot{\beta}}^\mu & -D \end{pmatrix}. \quad (6.22)$$

The bundle curvature $\Omega_{\pm}^{\mu\nu}$ given by Eq.(6.7) amounts to

$$\Omega_+^{\mu\nu} = \begin{pmatrix} -iF^{\mu\nu} & 0 & 0 \\ \frac{1}{\sqrt{2}}[\sigma_{E\alpha\dot{\alpha}}^\nu(\mathcal{D}^\mu\bar{\lambda}^{\dot{\alpha}}) - \sigma_{E\alpha\dot{\alpha}}^\mu(\mathcal{D}^\nu\bar{\lambda}^{\dot{\alpha}})] & -iF^{\mu\nu}\delta_\alpha^\beta & 0 \\ 0 & \frac{1}{\sqrt{2}}[(\mathcal{D}^\mu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}_E^{\nu\dot{\alpha}\beta} - (\mathcal{D}^\nu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}_E^{\mu\dot{\alpha}\beta}] & -iF^{\mu\nu} \end{pmatrix}, \quad (6.23)$$

$$\Omega_-^{\mu\nu} = \begin{pmatrix} -iF^{\mu\nu} & 0 & 0 \\ \frac{1}{\sqrt{2}}[\bar{\sigma}_E^{\nu\dot{\alpha}\alpha}(\mathcal{D}^\mu\lambda_\alpha) - \bar{\sigma}_E^{\mu\dot{\alpha}\alpha}(\mathcal{D}^\nu\lambda_\alpha)] & -iF^{\mu\nu}\delta_\alpha^{\dot{\beta}} & 0 \\ 0 & \frac{1}{\sqrt{2}}[(\mathcal{D}^\mu\lambda^\alpha)\sigma_{E\alpha\dot{\beta}}^\nu - (\mathcal{D}^\nu\lambda^\alpha)\sigma_{E\alpha\dot{\beta}}^\mu] & -iF^{\mu\nu} \end{pmatrix}. \quad (6.24)$$

From Eq.(6.21) we have

$$\begin{aligned} \text{Str } \mathbb{E}_+^2 &= \text{Tr} \left[D^2 - i\lambda^\beta \sigma_{E\beta\dot{\beta}}^\mu (\mathcal{D}_\mu \bar{\lambda}^{\dot{\beta}}) \right] \\ &\quad - \text{Tr} \left[-i\sigma_{E\alpha\dot{\alpha}}^\mu (\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}}) \lambda^\alpha - \sigma_{E\beta}^{\mu\nu\dot{\beta}} \sigma_{E\dot{\beta}}^{\lambda\kappa\alpha} F_{\mu\nu} F_{\lambda\kappa} + i\lambda_\alpha (\mathcal{D}_\mu \bar{\lambda}_{\dot{\beta}}) \bar{\sigma}_E^{\mu\dot{\beta}\alpha} \right] \\ &\quad + \text{Tr} \left[i(\mathcal{D}_\mu \bar{\lambda}_{\dot{\alpha}}) \bar{\sigma}_E^{\mu\dot{\alpha}\beta} \lambda_\beta + D^2 \right] \\ &= \text{Tr} \left[2D^2 - 4i\bar{\lambda}_{\dot{\beta}} \bar{\sigma}_E^{\mu\dot{\beta}\beta} (\mathcal{D}_\mu \lambda_\beta) - F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \varepsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa} \right]. \end{aligned} \quad (6.25)$$

Here the trace is taken over the 3×3 matrices of the color degrees of freedom. As for the antichiral sector given by Eq.(6.22) we have the following result:

$$\text{Str } \mathbb{E}_-^2 = \text{Tr} \left[2D^2 - 4i\bar{\lambda}_{\dot{\beta}} \bar{\sigma}_E^{\mu\dot{\beta}\beta} (\mathcal{D}_\mu \lambda_\beta) - F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \varepsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa} \right]. \quad (6.26)$$

Equation (6.25) and Eq.(6.26) give the following expression:

$$\begin{aligned} \text{tr}_V(\mathbb{E}^2) &= \text{Str } \mathbb{E}_+^2 + \text{Str } \mathbb{E}_-^2 \\ &= 2\text{Tr} \left[2D^2 - 4i\bar{\lambda}_{\dot{\beta}} \bar{\sigma}_E^{\mu\dot{\beta}\beta} (\mathcal{D}_\mu \lambda_\beta) - F_{\mu\nu} F^{\mu\nu} \right]. \end{aligned} \quad (6.27)$$

The supertrace of $\Omega_{\pm\mu\nu}\Omega_{\pm}^{\mu\nu}$ amounts to

$$\begin{aligned} \text{Str } \Omega_{\pm\mu\nu}\Omega_{\pm}^{\mu\nu} &= \text{Tr}[-F_{\mu\nu}F^{\mu\nu}] - \text{Tr}[-F_{\mu\nu}F^{\mu\nu}\mathbf{1}_2] + \text{Tr}[-F_{\mu\nu}F^{\mu\nu}] \\ &= 0. \end{aligned} \quad (6.28)$$

Now that we are ready to calculate the heat kernel coefficients. From Eq.(6.3) we obtain

$$a_0 = 0, \quad (6.29)$$

since the number of freedom of the bosonic sector is equal to the number of freedom of the fermionic sector due to the supersymmetry, so that $\text{Str } \mathbb{I} = 0$. Equation (6.29) indicates that the cosmological constant in the supersymmetric theory vanishes. The coefficient a_2 also vanishes,

$$a_2 = 0, \quad (6.30)$$

since D and $F_{\mu\nu}$ in $\text{tr}_V(\mathbb{E})$ are traceless 3×3 matrices with respect to the color degrees of freedom. Finally,

Eq.(6.27) and Eq.(6.28) give

$$\begin{aligned} a_4 &= \frac{1}{16\pi^2} \int_M dx^4 \\ &\quad \times \text{Tr} \left[2D^2 - 4i\bar{\lambda}_{\dot{\beta}} \bar{\sigma}_E^{\mu\dot{\beta}\beta} (\mathcal{D}_\mu \lambda_\beta) - F_{\mu\nu} F^{\mu\nu} \right], \end{aligned} \quad (6.31)$$

since $\text{tr}_V(\mathbb{E}_{;\mu}^\mu) = 0$.

The Euclidean super Yang-Mills action I_E is now given by

$$I_E = \text{Tr}_{L^2} f \left(\tilde{\mathcal{D}}_M^2 \right) = f_4 a_4. \quad (6.32)$$

In order to obtain the physical action we change the signature back to the Minkowskian, $\eta^{\mu\nu} \rightarrow g^{\mu\nu}$ and rescale the vector supermultiplet as $\{A_\mu, \lambda_\alpha, D\} \rightarrow \{g_c A_\mu, g_c \lambda_\alpha, g_c D\}$, where g_c turns out to be the gauge coupling constant. After this procedure we have the fol-

lowing super Yang-Mills action:

$$I_{SYM} = \int_M dx^4 \times \text{Tr} \left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i \bar{\lambda}_{\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\beta} (\mathcal{D}_{\mu} \lambda_{\beta}) + D^2 \right], \quad (6.33)$$

where we fixed the constant f_4 such that

$$\frac{f_4}{8\pi^2} = \frac{1}{g_c^2}, \quad (6.34)$$

and normalized the matrices t_{ℓ} in Eq.(5.27)–Eq.(5.29) as

$$\text{Tr}(t_{\ell} t_m) = 2 \delta_{\ell m}. \quad (6.35)$$

The final step is to derive the action integral including quarks. Here we work in the Minkowskian signature. Let us first define the supersymmetric product of two chiral

supermultiplets Ψ_+ and Ψ_-^{\dagger} by

$$(\Psi_-, \Psi_+) = \int_M dx^4 \Psi_-^{\dagger} \Gamma_0 \Psi_+, \quad (6.36)$$

where Γ_0 is the constant matrix such that

$$\begin{aligned} \Psi_-^{\dagger} \Gamma_0 \Psi_+ &= (\varphi_-^*, (\psi_-^*)^{\alpha}, F_-^*) \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \psi_{+\alpha} \\ F_+ \end{pmatrix} \\ &= \varphi_-^* F_+ + F_-^* \varphi_+ - (\psi_-^*)^{\alpha} \psi_{+\alpha}. \end{aligned} \quad (6.37)$$

This expression is the F-term of the product of two chiral supermultiplets and it transforms into a space derivative under the supersymmetry transformation given by Eq.(2.2). The supersymmetric product of two antichiral supermultiplets is also defined in the same way.

The kinetic parts of the actions for the chiral supermultiplet $\Psi_+ = (\varphi_+, \psi_{+\alpha}, F_+)$ and antichiral supermultiplet $\Psi_- = (\varphi_-, \psi_-^{\dot{\alpha}}, F_-)$ are obtained by the use of the following formulae:

$$\Psi_+^{\dagger} \Gamma_0 \tilde{\mathcal{D}} \Psi_+ = -\mathcal{D}_{\mu} \varphi_+^* \mathcal{D}^{\mu} \varphi_+ - i \psi_{+\dot{\alpha}}^* \bar{\sigma}^{\mu\dot{\alpha}\alpha} \mathcal{D}_{\mu} \psi_{+\alpha} - \sqrt{2}i \left(\varphi_+^* \lambda^{\alpha} \psi_{+\alpha} - \psi_{+\dot{\alpha}}^* \bar{\lambda}^{\dot{\alpha}} \varphi_+ \right) - \varphi_+^* D \varphi_+ + F_+^* F_+, \quad (6.38)$$

$$\Psi_-^{\dagger} \Gamma_0 \tilde{\mathcal{D}} \Psi_- = -\mathcal{D}_{\mu} \varphi_-^* \mathcal{D}^{\mu} \varphi_- - i \psi_-^* \sigma_{\alpha\dot{\alpha}}^{\mu} \mathcal{D}_{\mu} \psi_-^{\dot{\alpha}} - \sqrt{2}i \left(\varphi_-^* \bar{\lambda}_{\dot{\alpha}} \psi_-^{\dot{\alpha}} - \psi_-^* \lambda_{\alpha} \varphi_- \right) + \varphi_-^* D \varphi_- + F_-^* F_-, \quad (6.39)$$

where $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}$ are given by Eq.(4.42) and Eq.(4.43).

The wave function of quarks, Ψ consists of the chiral- and antichiral-supermultiplets defined by Eq.(5.20)–Eq.(5.22) and Eq.(5.24)–Eq.(5.26), respectively. They are denoted by

$$\Psi_L^a = (\tilde{q}_L^a, q_{L\alpha}^a, F_L^a), \quad (6.40)$$

$$\Psi_R^a = (\tilde{q}_R^a, q_R^{a\dot{\alpha}}, F_R^a), \quad (6.41)$$

where a is the color index. The spectral action principle asserts that the action for quarks is obtained by the total Dirac operator $\tilde{\mathcal{D}}_0$ so that

$$\begin{aligned} I_{\text{quark}} &= (\Psi, i\tilde{\mathcal{D}}_0 \Psi) \\ &= (\Psi, i\tilde{\mathcal{D}}_M \Psi) + (\Psi, \gamma_M \otimes \mathcal{D}_F \Psi). \end{aligned} \quad (6.42)$$

Here γ_M is given by Eq.(3.41).

The first term of Eq.(6.42) is the kinetic part of the action, which is given by

$$\begin{aligned} I_{\text{kinetic}} &= (\Psi, i\tilde{\mathcal{D}}_M \Psi) \\ &= (\Psi_L, \tilde{\mathcal{D}} \Psi_L) + (\Psi_R, \tilde{\mathcal{D}} \Psi_R). \end{aligned} \quad (6.43)$$

The second term of Eq.(6.42) turns out to be the mass term of quarks:

$$I_{\text{mass}} = (\Psi, \gamma_M \otimes \mathcal{D}_F \Psi). \quad (6.44)$$

As for the kinetic part of the action, using Eq.(6.38) and Eq.(6.39) we can confirm that

$$\begin{aligned} I_L &= (\Psi_L, \tilde{\mathcal{D}} \Psi_L) \\ &= \int_M dx^4 \left[-\mathcal{D}_{\mu} \tilde{q}_L^* \mathcal{D}^{\mu} \tilde{q}_L - i \bar{q}_{L\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \mathcal{D}_{\mu} q_{L\alpha} - \sqrt{2}i g_c \left(\tilde{q}_L^* \lambda^{\alpha} q_{L\alpha} - \bar{q}_{L\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \tilde{q}_L \right) - \tilde{q}_L^* D \tilde{q}_L + F_L^* F_L \right], \end{aligned} \quad (6.45)$$

for the left-handed quark supermultiplet and

$$\begin{aligned} I_R &= \left(\Psi_R, \tilde{\mathcal{D}} \Psi_R \right) \\ &= \int_M dx^4 \left[-\mathcal{D}_\mu \tilde{q}_R^* \mathcal{D}^\mu \tilde{q}_R - i \tilde{q}_R^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \mathcal{D}_\mu q_R^{\dot{\alpha}} - \sqrt{2} i g_c (\tilde{q}_R^* \bar{\lambda}_{\dot{\alpha}} q_R^{\dot{\alpha}} - \bar{q}_R^\alpha \lambda_\alpha \tilde{q}_R) + \tilde{q}_R^* D \tilde{q}_R + F_R^* F_R \right], \end{aligned} \quad (6.46)$$

for the right-handed quark supermultiplet.

Finally let us give the mass term of quarks. From Eq.(6.44) this part of the action amounts to

$$I_{\text{mass}} = (\Psi_L, im^T \Psi_R) - (\Psi_R, im \Psi_L), \quad (6.47)$$

where we recall that \mathcal{D}_F is given by Eq.(5.8). We redefine here the phase of Ψ_L as $\Psi_L \rightarrow i\Psi_L$, then we have

$$\begin{aligned} I_{\text{mass}} &= (\Psi_R, m \Psi_L) + \text{h.c.} \\ &= \int_M dx^4 \\ &\times [\tilde{q}_R^* m F_L + F_R^* m \tilde{q}_L - \bar{q}_R^\alpha m q_{L\alpha} + \text{h.c.}]. \end{aligned} \quad (6.48)$$

To summarize we have shown that the supersymmetric QCD action on the noncommutative geometry is obtained by the following formula of the spectral action principle:

$$I_{\text{SQCD}} = \text{Tr}_{L^2} \tilde{\mathcal{D}}_0^2 + \left(\Psi, i \tilde{\mathcal{D}}_0 \Psi \right), \quad (6.49)$$

where $\tilde{\mathcal{D}}_0$ is given by Eq.(6.2).

VII. CONCLUSIONS

In this paper we have studied how to introduce supersymmetry into the NCG models. In order to incorporate supersymmetry we enlarged the Hilbert space to

include bosons as well as fermions, which constitute a supermultiplet. Then we defined the generalized Dirac operator \mathcal{D}_M which operates on the supermultiplets in the Hilbert space. Although supersymmetry must be formulated in the Minkowskian signature, the axioms of NCG are stated in the Euclidean one. So, we changed the signature by the Wick rotation when we checked the axioms of NCG and calculated the spectral action.

A vector supermultiplet was introduced as the internal fluctuation on the metric. The modified Dirac operator $\tilde{\mathcal{D}}_M$ due to the fluctuation turned out to be supersymmetric and gauge-covariant if we introduce the internal degrees of freedom as the finite geometry. In this paper we considered the supersymmetric QCD model where the finite geometry is represented by the algebra of complex matrices.

Following the prescription of NCG we calculated the spectral action by using our generalized supersymmetric Dirac operator $\tilde{\mathcal{D}}_M$. We have found that the super Yang-Mills action was successfully derived based on the spectral action principle. The other parts of the action which include kinetic terms and mass terms of quarks and their superpartners were also obtained. As a result we found that the whole supersymmetric QCD action was expressed by the simple formula of the spectral action principle.

The method proposed in this paper to incorporate supersymmetry in NCG is applicable to other models such as the supersymmetric standard model. In this model the Higgs bosons and their superpartners are introduced as the internal fluctuation on the metric of the finite geometry. The detailed discussions and calculations on this subject will be given in a separate paper [28].

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